FINITE DIMENSIONAL REPRESENTATIONS OF ALGEBRAS

BY

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Notation. All rings considered will be rings with 1 and maps between such rings will be 1 preserving unless otherwise specified. If R is a ring and n a natural number, $(R)_n$ will denote the full ring of $n \times n$ matrices over R. If $\phi : R \to S$ is a map $(\phi)_n : (R)_n \to (S)_n$ is the naturally induced map on the entries.

1. Introduction

Let R be a ring, K a field, and n a natural number. We will be concerned with the following type of questions:

(a) Classify the representations of R in $(K)_n$ (or n dimensional representations).

(b) Classify n dimensional representations up to the natural equivalence. If $\phi_1, \phi_2 : R \to (K)_n$ we say ϕ_1 is equivalent to ϕ_2 if there is a K automorphism γ of $(K)_n$ such that $\gamma\phi_1 = \phi_2$.

(c) Classify equivalence classes of irreducible and semisimple representations.

For the sake of simplicity we will restrict ourselves mainly to the case that K is algebraically closed and R is a finitely generated K algebra. In fact, for most of this paper, we will assume that $R = K\{x_1, \cdots, x_n\}$ is a free algebra; at the end we will deduce from the results in this case the more general theorems for not necessarily free algebras.

Before describing the theorems that we will obtain, let us digress in order to motivate the use that we will make of the theory of rings with polynomial identities as a tool for attacking the problems stated before.

Let us recall, therefore, some of the basic structural results of this theory and interpret them in terms of representation theory.

THEOREM 1.1 (Kaplansky [4], [5]). *If R is a primitive ring satisfying a nontrivial polynomial identity of degree n then R is a central simple algebra of degree $\leqq n/2$.*

This theorem can be strengthened considerably with further restrictions on R.

THEOREM 1.2 (Refer to [5]). *Under the hypotheses of Theorem* 1.1 *if, furthermore, R is finitely generated over K and K is algebraically closed then* $R \simeq (K)_m$, $m \leqq n/2$.

Let us define a map $\phi : R \to (L)_h$, where L is a field, to be an irreducible representation of dimension h, if $L\phi(R) = (L)_h$. If L is algebraically closed this is equivalent to saying that $\phi(R)$ is an irreducible set of matrices (hence the name irreducible representation).

THEOREM 1.3 ([4], [5]). *A ring R possesses an injective irreducible representation of dimension h if and only if R is a prime ring satisfying a polynomia identity and $2h$ is the minimum degree of the identities satisfied by R.*

This theorem is in fact a consequence of a stronger result. Let us say that a prime ring R has degree h if $2h$ is the minimum degree of any polynomial identity satisfied by R, (this minimum degree is known to be even if it exists). If R does not satisfy any identity we do not give a degree to R.

THEOREM 1.4 (Posner [4], [5]). *If R is a prime ring of degree h, then R has a total ring of left and right quotients $Q(R)$ which is a simple algebra of dimension h^2 over its center.*

Let us now consider a special class of rings with polynomial identities which is particularly useful in representation theory.

DEFINITION 1.5. We say that a ring R satisfies the identities of $n \times n$ matrices if it satisfies all the polynomial identities with integer coefficients satisfied by $(Z)_n$ (Z denoting the ring of integers).

THEOREM 1.6 (M. Artin [2], [5]). *Let R be a ring satisfying all polynomial identities of $n \times n$ matrices. If R does not possess any representation of dimension $\leqq n-1$ then it is an Azumaya algebra of constant rank n^2 over its center.*

The converse of this theorem is also true and essentially trivial.

Finally we will need a basic tool in the theory which is the existence of n central polynomials.

DEFINITION 1.7. A non-commutative polynomial $f(x_1, \cdots, x_h) \in \Lambda\{x_1, \cdots, x_h\}$ (Λ a commutative ring) is said to be n-central if, whenever computed in $(A)_n$, A a commutative Λ-algebra, it yields values in the center A of $(A)_n$.

The main existence theorem for n-central polynomials is the following.

THEOREM 1.8 (Formanek [3], [5]). *There exists for every n an n-central* *polynomial $f(x_1,\cdots,x_h)$ with integer coefficients, multilinear in all but one variable, without constant term which is not identically zero when computed in $(A)_n$, A any commutative ring.*

2. n-dimensional representations

Following the brief summary, given in the introduction, of the main theorems on rings with polynomial identities we shall now start working on the main topic of this paper.

Let us denote by $S = K\{x_1,\cdots,x_h\}$ the free algebra in h variables. To give a representation $\phi : S \to (K)_n$ is equivalent to giving the h matrices $(\phi(x_1), \phi(x_2),\cdots,\phi(x_h))$. Therefore the set of maps $\mathscr{M}(S,(K)_n)$ can be identified to the affine space $(K)_n^h = K^{hn^2}$ of h tuples of $n \times n$ matrices. We have an evaluation map $S \times \mathscr{M}(S,(K)_n) \to (K)_n$ given by the formula $(a,\phi) \to \phi(a)$. If we let ϕ vary in $(K)_n^h$ and fix $a \in S$ we obtain a map $\bar{a} : (K)_n^h \to (K)_n$ which is easily verified to be a polynomial map. Let us indicate by $\mathscr{P}((K)_n^h, (K)_n)$ the set of polynomial maps from the vector space $(K)_n^h$ to $(K)_n$. Since $(K)_n$ is a ring, $\mathscr{P}((K)_n^h, (K)_n)$ is also a ring by pointwise addition and multiplication of maps. This ring is easily seen to be isomorphic to the ring $(K[\xi_{s,t}^i])_n$ of $n \times n$ matrices over the ring $K[\xi_{s,t}^i] \simeq \mathscr{P}((K)_n^h, K)$ of polynomials in the variables $\xi_{s,t}^i$, $i = 1,\cdots,h$ and $s,t = 1,\cdots,n$. ($\xi_{s,t}^i$ is the coordinate map which consists of taking the s,t entry of the ith component matrix of an h-tuple of matrices).

Having made this identification, the map $a \to \bar{a}$ from S to $\mathscr{P}((K)_n^h, (K)_n)$ is a ring map

$$\pi : K\{x_1,\cdots,x_n\} \to (K[\xi_{s,t}^i])_n.$$

Under π the variable x_i goes to the matrix ξ_i whose s,t entry is $\xi_{s,t}^i$. We will refer sometimes to the matrices $\xi_1,\cdots,\xi_h$ as *the generic matrices over* K and to the ring $\bar{S} = \pi(S)$ as the *ring of generic matrices*. The map π is universal among maps in $n \times n$ matrices over commutative rings in the following way.

Given a map $\phi : K\{x_1,\cdots,x_h\} \to (A)_n$, A a commutative ring, there exists a unique map $\bar{\phi} : K[\xi_{s,t}^i] \to A$ such that the following diagram is commutative.

$$\begin{array}{ccc} K\{x_1,\cdots,x_n\} & \xrightarrow{\pi} & (K[\xi_{s,t}^i])_n \\ & \phi \searrow \quad \swarrow (\bar{\phi})_n & \\ & (A)_n & \end{array}$$

S is known to be an Öre domain; if D denotes its skew field of fractions then, as soon as $h \geqq 2$, D is a central simple algebra of degree n and there exists a unique map $\psi : D \to (K(\xi^i_{s,t}))_n$ such that the following diagram is commutative.

$$\begin{array}{ccc} S & \longrightarrow & (K[\xi^i_{s,t}])_n \\ \downarrow & & \downarrow \\ D & \longrightarrow & (K(\xi^i_{s,t}))_n \end{array}$$

Furthermore, if Z is the center of D, $\psi(Z)$ is contained in the center $K(\xi^i_{s,t})$ of $(K(\xi^i_{s,t}))_n$ and we have a canonical isomorphism

$$(K(\xi^i_{s,t}))_n \simeq D \otimes_Z K(\xi^i_{s,t}) \text{ (see[5])}.$$

Consider now the group G of K automorphisms of $(K)_n$. G is isomorphic to the projective linear group $Pl(h, K)$ since every automorphism of $(K)_n$ is inner.

G acts clearly on the set $\mathcal{M}(S, (K)_n)$ of representations and its orbits are exactly the equivalence classes of representations considered in the introduction. Therefore to classify the representations of S up to equivalence is essentially the study of the orbit space of the affine space $(K)^h_n$ under G.

Now G is an algebraic group and the action of G on the affine space $(K)^h_n$ is algebraic; it makes sense, therefore, to look for an algebraic variety as quotient. Now it is clear that some orbits of G are not closed; therefore a quotient variety, even if it exists, will not classify the orbit space.

We will go back to this analysis in a moment but first we want to make a further remark about the action of G. G acts on $\mathcal{P}((K)^h_n, (K)_n)$ in the following way: if $f : (K)^h_n \to (K)_n$ is a polynomial function and $g \in G$, we set $(gf)(p) = g(f(g^{-1}(p)))$. G gives in this way a group of automorphisms of the ring $\mathcal{P}((K)^h_n, (K)_n)$ and it is easy to verify that the ring of generic matrices is formed of invariant elements under G. Clearly this G action extends to an action of G on $(K(\xi^i_{s,t}))_n$. It will follow from our further work that D is exactly the ring of invariants of this G action.

Let us go back now to the study of the orbits of G acting on $(K)^h_n$.

If $\phi : S \to (K)_n$ is a representation (that is, a point of $(K)^h_n$) then clearly K^n becomes an S module via ϕ. If K^n, as an S module via ϕ, is completely reducible we will say that ϕ is semisimple. In general, let $K^n = V_0 \supset V_1 \supset \cdots \supset V_t = 0$ be a composition series of this module. Then $\oplus V_i/V_{i+1} = W$ is a completely reducible

S module of dimension n over K. If we choose a basis of W we deduce, from this module structure, an n-dimensional representation $\phi^s : S \to (K)_n$. This representation is clearly dependent on the choice of a basis of W. On the other hand, different choices of base give equivalent representations. Most important of all, this representation does not depend, up to equivalence, on the composition series of V chosen. This is a restatement of the Jordan-Hölder theorem in this case. We have thus found, intrinsically associated to ϕ, an equivalence class of semisimple representations. We will denote (somewhat imprecisely) by ϕ^s any semisimple representation in this class.

THEOREM 2.1 (M. Artin [2]). *ϕ^s is in the closure of the orbit of ϕ.*

This result implies that we will not be able, in any type of quotient variety of $(K)_n^h$ by G, to distinguish between ϕ and ϕ^s. Therefore this theorem motivates our future work which will consist in constructing a variety whose points over K correspond to the equivalence classes of semisimple representations of S in $(K)_n$.

3. The ring C_n

We have seen, in the previous paragraph, that G acts on the ring $(K[\xi_{s,t}^i])_n$ and that the ring of generic matrices is formed of invariant elements. Since G acts as a group of automorphisms of $(K[\xi_{s,t}^i])_n$, the center $K[\xi_{s,t}^i]$ of this ring is stable under this G action. $K[\xi_{s,t}^i]$ is the ring of polynomial functions on $(K)_n^h$ and an element $f \in K[\xi_{s,t}^i]$ is invariant under G if and only if it is constant on the orbits. Let us denote by I_n the ring of invariants of $K[\xi_{s,t}^i]$ under G. Our first step will be to find a large subring C_n of I_n, made of a simple type of invariants on which we will be able to operate. Now if $a \in S$, $\pi(a) \in (K[\xi_{s,t}^i]_n)$ is invariant under the action of G; therefore it is clear that the coefficients of the characteristic polynomial of the matrix $\pi(a)$ are invariant under G, that is, they belong to I_n. Let C_n be the subring of I_n generated by these elements.

PROPOSITION 3.1. *C_n is a finitely generated K algebra.*

Before starting the proof of this theorem we need to recall another important theorem from the theory of rings with polynomial identities.

THEOREM 3.2 (Širšov [6]). *Let R be an algebra over a commutative ring Λ satisfying a power of the standard identity $S_{2k}(x_1, \cdots, x_{2k})^h$ as a stable identity (that is, $R \otimes A$ satisfies it for all A commutative). Assume that $R = \Lambda\{r_1, \cdots, r_h\}$ is finitely generated and that the monomials in the r_i's of degree $\leq k^2$ are integral over Λ; then R is a finite Λ module.*

We go back now to 3.1.

PROOF OF PROPOSITION 3.1. Let C'_n be the subring of C_n generated by the coefficients of the characteristic polynomials of the monomials of degree $\leqq k^2$ in the elements $\xi_i = T(x_i)$. Consider the ring $C'_n\{\xi_1, \cdots, \xi_h\} \supseteq (K[\xi^i_{\cdot,t}])_n$. The hypotheses of Theorem 3.2 are satisfied since each monomial in the ξ_i's of degree $\leqq n^2$ satisfies its characteristic polynomial which, by the construction of C'_n, is a monic polynomial with coefficients in C'_n. Therefore $C'_n\{\xi_1, \cdots, \xi_h\}$ is a finite C'_n module. This implies that if $a \in K\{x_1, \cdots, x_h\}$, the element $\pi(a)$ is integral over C'_n. Therefore the coefficients of the characteristic polynomial of $\pi(a)$ are integral over C'_n and so C_n is integral over C'_n. Now C_n is contained in the integral closure of C'_n in $K[\xi^i_{s,t}]$ and, by commutative ring theory, one knowns that this integral closure is a finite C'_n module. Therefore C_n is a finitely generated K algebra.

4. The variety associated to C_n

Let $V = V(C_n)$ be the affine variety associated to C_n and denote by U, for simplicity, the affine space $(K)^h_n$ of n dimensional representations of $K\{x_1, \cdots, x_h\}$ in $(K)_n$.

We have a canonical map $p : U \to V$ and we claim that:

(a) $p(\phi) = p(\psi)$ if and only if $\phi^s = \psi^s$;

(b) p is an onto map.

These two facts together will show the following.

THEOREM 4.1. *The points of $V(C_n)$ are in one-to-one correspondence with the set of equivalence classes of semisimple representations under the natural map $[\phi] \to p(\phi)$ ($[\phi]$ denoting the equivalence class of a semisimple representation ϕ).*

PROOF. (a) Let $\phi : K\{x_1, \cdots, x_n\} \to (K)_n$ be a representation. If $a \in \ker \phi$ then $\phi(a) = 0$ so the characteristic polynomial $\chi_{\bar{a}}(x)$ of $\bar{a} = \pi(a)$ is mapped under the classifying map $\bar{\Phi}$ of ϕ

$$\begin{array}{ccc} K\{x_1, \cdots, x_h\} & \xrightarrow{\pi} & (K[\xi^i_{s,t}])_n \\ & \phi \searrow \quad \swarrow (\bar{\Phi})_n & \\ & (K)_n & \end{array}$$

into $x^n = \chi_0(x)$. Conversely, let ϕ be semisimple and I an ideal of $K\{x_1, \cdots, x_h\}$ made of elements a such that the characteristic polynomial of $\bar{a}$ is mapped to x^n

under $\bar{\Phi}$. Then $\phi(I)$ is made of nilpotent elements and so, being ϕ semisimple, $\phi(I) = 0$. Therefore $\ker\phi$ is identified by the restriction of the classifying map $\bar{\Phi}$ to C_n. Now if $M = \ker\phi$, we have $K\{x_1, \cdots, x_n\}/M$ is a semisimple algebra isomorphic to $\oplus\,(K)_{h_i}$; for some h_i's, ϕ is identified up to equivalence by saying how many times the irreducible representation associated to the ith summand appears in ϕ. This number is identified, on the other hand, by the rank of $\phi(l_i)$, l_i being the unit of $(K)_{h_i}$; this rank is the number of eigenvalues equal to 1 of the matrix $\phi(l_i)$ and therefore is identified by the characteristic polynomial of $\phi(l_i)$. This finishes (a) having shown ϕ is completely identified, up to equivalence, by the restriction to C_n of the classifying map $\bar{\Phi}$.

(b) Consider the ring $R = C_n\{\xi_1, \cdots, \xi_h\}$; as in Proposition 3.1, we know that R is a finite C_n module. Let $\lambda : C_n \to K$ be a point in $V = V(C_n)$. Consider the algebra $\bar{R} = R\otimes_\lambda K$. $\bar{R}$ is a finite dimensional K algebra; let J be its radical and $R^* = \bar{R}/J$. $R^* \simeq \oplus_{i=1}^{t}(K)_{h_i}$ for some h_i's. Consider the determinant map $\det : (K[\xi^i_{s,t}])_n \to K[\xi^i_{s,t}]$. We may remark simply that $\det(R) \subseteq C_n$, since $\det(\sum \lambda_i a_i)$, $\lambda_i \in C_n$, $a_i \in K\{\xi_1, \cdots, \xi_h\}$ is a polynomial in the λ_is whose coefficients are easily seen to be in C_n. The same remark shows that if $\mathcal{M} = \ker\lambda$ and $a = a' + \sum m_i a_i$, $m_i \in \mathcal{M}$, we have $\det(a) = \det(a') + n$, $n \in \mathcal{M}$. These remarks show that the polynomial map det factors and gives a polynomial map N of degree n such that the following diagram is commutative.

$$\begin{array}{ccc} R & \xrightarrow{\det} & C_n \\ \downarrow & & \downarrow \\ \bar{R} & \xrightarrow{N} & K \end{array}$$

N is clearly multiplicative and of degree n since det is so ; furthermore if we consider the characteristic polynomial $\chi_a(x)$ of an element $a \in R$, we have that $\chi_a(x) = \det(x - a)$ maps, under λ, to $N(x - \bar{a})$ and thus $\bar{a}$ satisfies the polynomial $N(x - \bar{a})$ which we will denote by $\bar{\chi}_{\bar{a}}(x)$.

We want now to show that N factors through R^*. Let $u \in J$; since N is multiplicative we have $N(u) = 0$. Furthermore, we claim that if $r \in \bar{R}$ and $u \in J$, we have $N(r) = N(r + u)$. It is sufficient to show this for the invertible elements r of $\bar{R}$ since they form a Zariski open set of $\bar{R}$, which is a finite dimensional vector space over K, and thus an affine variety for which N is an algebraic map. If r is invertible, $N(r + u) = N(r) \cdot N(1 + r^{-1}u)$, therefore it is sufficient to show that for any

$u \in J$, $N(1 + u) = 1$. Now the set $1 + J$ is an algebraic group with a composition series whose factors are isomorphic to K^{ad}, therefore, since N is a morphism $1 + J \to K^*$ of algebraic groups we deduce that N is constant and equal to 1. These facts show that N factors to give a new multiplicative map, which we still denote N, $N : R^* \to K$ of degree n. Furthermore it is clear that, if $r \in R^*$, $N(x - r)$ is a polynomial satisfied by r.

Consider now the semisimple algebra R^*, $R^* \simeq \oplus_{i=1}^r (K)_{h_i}$. Let us denote by $\tilde{R}$ the group of invertible elements of R^*. $\tilde{R}$ is isomorphic to $\Pi_{i=1}^r Gl(h_i, K)$, $Gl(h_i,K)$ being the multiplicative group of $(K)_{h_i}$.

The map N, restricted to $Gl(h_i, K)$, gives a morphism $N_i : Gl(h_i, K) \to K^*$. Now the map $\det : Gl(h_i, K) \to K^*$ is the universal map in commutative algebraic groups and so N_i factors as $N_i : Gl(h_i, K) \xrightarrow{\det} K^* \xrightarrow{\psi_i} K^*$. ψ_i is a morphism of K^* in K^* so there is a $t_i \in Z$ such that $\psi_i(a) = a^{t_i}$ for all $a \in K^*$; we have therefore the following formula for N restricted to $\tilde{R}$:

$$N((a_1, a_2, \cdots, a_r)) = \Pi \det (a_i)^{t_i}$$

for $(a_1, a_2, \cdots, a_r) \in \Pi_{i=1}^r Gl(h_i, K) = \tilde{R}$. Let us now remember that N extends to the whole space R^*; this, of course, implies that all the t_i's are $\geqq 0$ and N is given by the same formula. Finally, since every element a of R^* satisfies the polynomial $N(x - a)$, it follows that $t_i > 0$ for all i; using the fact that N is of degree n we have $\Sigma_{i=1}^r t_i n_i = n$. Let now γ_i be the irreducible representation of R^* associated to its ith factor $(K)_{h_i}$. We construct the representation $\gamma = \Sigma\, t_i \gamma_i$. Since γ_i is of dimension h_i it follows that γ is of dimension $\Sigma\, t_i h_i = n$. Choose an explicit representation in this equivalence class, $\bar{\gamma} : R^* \to (K)_n$; by the construction of $\bar{\gamma}$ and the formula $N(a) = \Pi \det(a_i)^{t_i}$ $(a = (a_1, a_2, \cdots, a_r))$ we have, for all $a \in R^*$, $N(a) = \det(\bar{\gamma}(a))$. Composing $\bar{\gamma}$ with the projection $\rho : R \to R^*$ we have a semisimple representation $\bar{\gamma}\pi : R \xrightarrow{\rho} R^* \xrightarrow{\bar{\gamma}} (K)_n$ such that $\bar{\gamma}$ $(\det(r)) = \lambda \det(r)) = N(\rho(r)) = \det(\bar{\gamma}(r))$. Let $\gamma^* : K[\xi^i_{s,t}] \to K$ be the classifying map of $\bar{\gamma}\pi$, so that the following diagram is commutative.

$$\begin{array}{ccc} K\{x_1, \cdots, x_n\} & \xrightarrow{\pi} & (K[\xi^i_{s,t}])_n \\ & \bar{\gamma}\pi \searrow \quad \swarrow (\gamma^*)_n & \\ & (K)_n & \end{array}$$

If $a \in K\{x_1, \cdots, x_h\}$ we have just seen that $\gamma^*(\det(\pi(a))) = \lambda(\det(\pi(a)))$; more generally, we will have $\gamma^*(\det(x - \pi(a))) = \lambda(\det(x - \pi(a)))$, hence λ and γ^*

coincide on the ring C_n. We have thus found a point in the variety U which maps to the point of V corresponding to λ, so the map $p : U \to V$ is onto as announced.

We want to discuss now the theory of representations for a general finitely generated algebra $R = K\{r_1,\cdots,r_h\}$. R can be considered as a quotient of the free algebra $S = K\{x_1,\cdots,x_h\}$, $R \simeq S/I$. Consider $\pi(I) \subseteq (K[\xi^i_{s,t}])_n$ and let $(J)_n$ be the ideal of $(K[\xi^i_{s,t}])_n$ generated by $\pi(I)$. The map $S \to (K[\xi^i_{s,t}])_n \to (K[\xi^i_{s,t}]/J)_n$ factors through I giving rise to a map $\bar{\pi}$ making the following diagram commutative.

$$\begin{array}{ccc} S & \xrightarrow{\pi} & (K[\xi^i_{s,t}])_n \\ \downarrow & & \downarrow \\ R & \xrightarrow{\bar{\pi}} & (K[\xi^i_{s,t}]/J)_n \end{array}$$

$\bar{\pi}$ has the same kind of universal property for R that π had for S; if $\gamma : R \to (A)_n$ is a map in the ring of $n \times n$ matrices over a commutative ring A, there exists a unique map $\bar{\gamma} : K[\xi^i_{s,t}]/J \to A$ making the following diagram commutative.

$$\begin{array}{ccc} R & \xrightarrow{\bar{\pi}} & (K[\xi^i_{s,t}]/J)_n \\ {\scriptstyle\gamma}\searrow & & \swarrow{\scriptstyle(\bar{\gamma})_n} \\ & (A)_n & \end{array}$$

This property implies, immediately, that the ring $V_n(R) = K[\xi^i_{s,t}]/J$ is uniquely determined, does not depend on the presentation of R, and is a functor of R.

Now we construct, as we did for the free algebra, the ring $C_n(R)$ generated by the coefficients of the characteristic polynomials of the elements of $\bar{\pi}(R) \subseteq (V_n(R))_n$.

$C_n(R)$ is readily seen to be a functor of R with the further property that given an onto map $R \to R'$ the induced map $C_n(R) \to C_n(R')$ is also onto. Thus if R is presented as S/I, $S = K\{x_1,\cdots,x_h\}$ we have an onto map $C_n(S) = C_n \to C_n(R)$. Consider now the subvariety W of V image of the variety associated to $V(C_n(R))$, we claim that this subvariety is formed exactly of the equivalence classes of the semisimple representations $S \to (K)_n$ which factor through I to give representations of R.

Now, if $x: R \to (K)_n$ is a representation, then by universality of $\bar{\pi}$, ψ is classified by a map $\psi : V_n(R) \to K$ such that the following diagram commutes.

$$\begin{array}{ccc} S & \longrightarrow & (K[\xi^i_{s,t}])_n \\ \downarrow & & \downarrow \\ R & \xrightarrow{\bar{\pi}} & (V_n(R))_n \\ & \psi \searrow & \downarrow \\ & & (K)_n \end{array}$$

Therefore the point of V associated with the composition $S \to R \xrightarrow{\psi} (K)_n$ lies in W. Conversely, consider a semisimple representation $\psi : S \to (K)_n$ whose associated point lies in W. We must show that $\psi(I) = 0$; since $\psi(S)$ is semisimple, this is equivalent to showing that the characteristic polynomial of any element of $\psi(I)$ is x^n. Let $\psi^* : C_n \to K$ be the map classifying ψ. Since ψ is in W, ψ^* factors through $C_n(R)$. Now if $a \in I$, in the map $C_n \to C_n(R)$ the characteristic polynomial of a is mapped in x^n so the theorem is fully proved.

5. Irreducible representations

We want to study now the irreducible representations of a ring R. For this purpose we will define some new functors on the category of rings. Let us denote by υ_n the variety of all rings satisfying the identities of $n \times n$ matrices. If R is any ring, then there is an ideal I minimal with the property that $R/I \in \upsilon_n$; calling $R^{(n)}$ the ring R/I, we define in this way a functor from the category $\mathscr{C}$ of rings to υ_n, adjoint to the inclusion $\upsilon_n \to \mathscr{C}$.

If $S \in \upsilon_n$ and $f(x_1, \cdots, x_k)$ is an n-central polynomial (refer to Definition 1.7), then any evaluation of f in S yields an element of the center of S. Let us consider the set of all elements of S obtained by evaluation of n-central polynomials subject to the restriction of being linear in at least one variable (as functions); call $F_n(S)$ this set. We easily have the following proposition.

PROPOSITION 5.1. (a) *$F_n(S)$ is an ideal of the center of S.*

(b) *$F_n(S)$ is a functor of S.*

(c) *If $S \to S'$ is onto so is $F_n(S) \to F_n(S')$.*

(d) *If $S \supseteq S'$ and $S' = SA$, A the center of S', then $F_n(S') = F_n(S)A$.*

We start now to analyze more closely $F_n(S)$.

PROPOSITION 5.2. *If $A \subset (K)_n$ is a proper subalgebra, then $F_n(A) = 0$.*

PROOF. First of all let us show that $F_n((K)_h) = 0$ for $h < n$. In fact we can embed $(K)_h$ in $(K)_n$ $(h < n)$ without preserving 1. If we evaluate an n central polynomial without constant term in $(K)_h \subseteq (K)_n$ we must obtain a central element of $(K)_n$, that is, a scalar; on the other hand we have an element of $(K)_h$. Now the only scalar contained in $(K)_h$ is 0 so this first fact is established. Now if $A \subseteq (K)_n$ and $A \neq (K)_n$ let J be the radical of A. We can assume K algebraically closed and so $A/J \simeq \oplus (K)_{h_i}$, $h_i < n$. Therefore $F_n(A/J) = 0$ and so $F_n(A) \subseteq J$; now J is made of nilpotent elements and the only scalar which is nilpotent is 0, so $F_n(A) = 0$.

COROLLARY 5.3. *A map* $\phi : R \to (K)_n$ *is an irreducible representation if and only if* $\phi(F_n(R)) \neq 0$.

PROOF. The map ϕ is irreducible if and only if $\phi(R)K = (K)_n$. This is true, by the previous proposition, if and only if $F_n(\phi(R)K) \neq 0$ but $F_n(\phi(R)K) = \phi(F_n(R))K$ and so the corollary is proved.

THEOREM 5.4. *A ring* $R \in \mathfrak{v}_n$ *is a rank* n^2 *Azumaya algebra over its center if and only if* $I \in F_n(R)$.

PROOF. By Theorem 1.6, R is a rank n^2 Azumaya algebra over its center if and only if it has no representation of dimension $\leqq n-1$. Now if R has a representation $\gamma : R \to (K)_h$, $h < n$, we have $\gamma(F_n(R)) \subseteq F_n((K)_h) = 0$, so $1 \notin F_n(R)$. Conversely, if R is an Azumaya algebra of rank n^2 over its center A, assume, by contradiction, that $1 \notin F_n(R)$. Since $F_n(R)$ is an ideal of A there is a maximal ideal M of A containing $F_n(R)$. R/MR is a simple algebra of degree n and so $F_n(R/MR) \neq 0$; on the other hand $F_n(R)$ should map onto $F_n(R/MR)$, a contradiction.

This theorem has various interesting corollaries.

COROLLARY 5.5. *If* $R \in \mathfrak{v}_n$ *has center* A *then*:

(a) *If* $S \subseteq A$ *is a multiplicative set and* $S \cap F_n(R) \neq \emptyset$ *then* R_S *is a rank* n^2 *Azumaya algebra over* A_S;

(b) *If* p *is a prime ideal of* A *then* R_p *is a rank* n^2 *Azumaya algebra over* A_p *if and only if* $p \not\supseteq F_n(R)$.

PROOF. All statements are obvious except, perhaps, the only if part of (b). Now if R_p is a rank n^2 Azumaya algebra over A_p we have $F_n(R_p) = A_p$. On the other hand, by Proposition 5.1 (d), we have $F_n(R_p) = F_n(R)A_p$ and so (b) follows completely.

REMARK. The set of prime ideals p of A such that $p \not\supseteq F_n(R)$ is an open set of Spec A.

COROLLARY 5.6. *Let $\psi : R_1 \to R_2$ be a map between two rank n^2 Azumaya algebras with centers W_1, W_2 respectively. Then $\psi(W_1) \subseteq W_2$ and so $R_2 \simeq R_1 \otimes_\psi W_2$.*

PROOF. Trivial since $W_i = F_n(R_i)$.

Let us now go back to the universal map $\pi_R : R \to (V_n(R))_n$ in $n \times n$ matrices over commutative rings.

PROPOSITION 5.7. *If $S \subseteq F_n(R)$ is a multiplicative set, $\pi_R(S) \subseteq V_n(R)$ and the localized map $R_S \to (V_n(R)_{\pi_R(S)})_n$ is the universal map of R_S.*

PROOF. We know that $\pi_R(F_n(R)) \subseteq F_n((V_n(R))_n) = V_n(R)$, so the first assertion is verified. Let now $\psi : R_S \to (A)_n$ be any map, A a commutative ring. We must show that there is a unique map $\bar{\psi} : V_n(R)_{\pi_R(S)} \to A$ such that the following diagram is commutative.

$$\begin{array}{ccc} R_S & \longrightarrow & (V_n(R)_{\pi_R(S)})_n \\ & \searrow \quad \swarrow & \\ & (A)_n & \end{array}$$

Now, by the universal property of π_R, there is a unique map $\psi' : V_n(R) \to A$ making the following diagram commutative.

$$\begin{array}{ccc} R & \longrightarrow & (V_n(R))_n \\ \searrow & & \\ R_S & & \Big\downarrow (\psi')_n \\ \psi \searrow & & \\ & (A)_n & \end{array}$$

The universal properties of localization allows us to factor ψ' through $V_n(R)_{\pi_R(S)}$ and so prove the proposition.

PROPOSITION 5.8. *If R is a rank n^2 Azumaya algebra over its center A and $A \to B$ is a map of commutative rings then $V_n(R \otimes_A B)$ is canonically isomorphic to $V_n(R) \otimes_A B$.*

PROOF. We have a map $R \otimes_A B \to (V_n(R) \otimes_A B)_n$ and we must show that it is universal. If $\psi : R \otimes_A B \to (U)_n$ is any map, U a commutative ring, then ψ induces a map $\psi_1 : R \to (U)_n$ and, by Corollary 5.6, a map $\psi_2 : B \to U$. Therefore, by the universal properties of V_n and of tensor product, we have the desired result.

To finish this preliminary analysis we have to study $V_n((A)_n)$ for a commutative ring A. For this purpose let us note that the functor $A \to \mathrm{Aut}_A(A)_n$ (the group of A automorphisms of the ring $(A)_n$) is representable by an affine group scheme of finite type over Z. Let us call the coordinate ring of this group scheme A_n. To give a map $\psi : (A)_n \to (U)_n, U$ a commutative ring, is equivalent to giving, by (5.6), a pair ψ_1, ψ_2 where $\psi_1 : A \to U$ is a map and ψ_2 is a U automorphism of $(U)_n$. Therefore the rings $V_n((A)_n)$ and $A \otimes_Z A_n$ represent the same set-valued functor on the category of commutative rings, so they are canonically isomorphic and it is immediately verified that under this isomorphism the canonical map $A \to V_n((A)_n)$ becomes the map $a \to a \otimes 1$ of A in $A \otimes_Z A_n$.

We are almost ready to collect all our information; we need only two simple remarks. First of all, if R is a rank n^2 Azumaya algebra over its center A we claim that $C_n(R) = A$. In fact the map $\pi_R : R \to (V_n(R))_n$ is an inclusion since it is known that there are inclusion maps of R in $n \times n$ matrices over commutative rings and π_R is universal. Therefore $A \subseteq V_n(R)$; now if $r \in R$ it is known that the characteristic polynomial of r has coefficients in A, so $C_n(R) \subseteq A$; on the other hand the Trace function Tr : $R \to A$ is onto and so $A \subseteq C_n(R)$.

Finally, consider for any $R \in \mathfrak{v}_n$ the map $\pi_R : R \to (V_n(R))_n$; if $r \in F_n(R)$ then $\pi_R(r)$ is a scalar so $\det \pi_R(r) = \pi_R(r)^n$; therefore if $r \in F_n(R)$, $\pi_R(r^n) \in C_n(R)$, call Φ the set of such elements of $C_n(R)$. An immediate consequence of Corollary 5.3 is the following proposition. Here R is a finitely generated K algebra, K algebraically closed.

PROPOSITION 5.9. *A map $\phi : C_n(R) \to K$ corresponds to an irreducible representation if and only if $\phi(\Phi) \neq 0$.*

PROOF. Obvious.

We are ready now to discuss the main theorem on irreducible representations. We will find it convenient to use the language of schemes. Let us indicate by V_R the open subscheme of Spec $V_n(R)$ where Φ does not vanish identically; similarly, U_R is the open subscheme of Spec $C_n(R)$ where Φ does not vanish. If R is a finitely generated K algebra, K algebraically closed, we know that, the geometric points in K of V_R correspond to the irreducible representations of R in $(K)_n$ and the points in K of U_R correspond to the equivalence classes of such representations. Let us indicate by $\mathscr{A}_n = \mathrm{Spec}\, A_n$, $\mathscr{A}_n$ is an affine group scheme. Clearly $\mathscr{A}_n$ acts on Spec $V_n(R)$ and also on V_R and the action is compatible with the projection $p : V_R \to U_R$; we have the following theorem.

THEOREM 5.10. *The space V_R with the canonical $\mathscr{A}_n$ action and projection* *p is a principal bundle with group $\mathscr{A}_n$ and base U_R. This bundle is locally trivial in the étale topology.*

PROOF. The statement is local in the Zariski topology so let $S \subseteq \Phi$ be multiplitive and let us look at $p_S : V_{R_S} \to U_{R_S}$. We can study the problem in this case. Now R_S is a rank n^2 Azumaya algebra over its center A_S. Consider an étale covering of $\operatorname{Spec} A_S$ by $\operatorname{Spec} A_\alpha$ such that $R_S \otimes_{A_S} A_\alpha$ is isomorphic to $(A_\alpha)_n$. By Proposition 5.8 we have that $V_{R_S} \times_{\operatorname{Spec} A_S} \operatorname{Spec} A_\alpha \simeq V_{(A_\alpha)_n}$; on the other hand $\operatorname{Spec} A_S = U_{R_S}$ and $V_{(A_\alpha)_n} \simeq \operatorname{Spec} A_\alpha \times \mathscr{A}_n$. One readily checks that the identification $V_{R_S} \times_{\operatorname{Spec} A_S} \operatorname{Spec} A_\alpha \simeq \operatorname{Spec} A_\alpha \times \mathscr{A}_n$ is compatible with the canonical $\mathscr{A}_n$ structure on the second factor and that the projection p_S becomes just the projection on the first factor. This proves the theorem in full.

Let us apply, finally, this theorem to the representations of the free algebra $S = K\{x_1, \cdots, x_n\}$.

COROLLARY 5.11. *The variety U_S of irreducible n dimensional representations of S is a smooth variety of dimension $h \cdot n^2 - (n^2 - 1)$.*

PROOF. V_S is an open set of the affine space K^{hn^2}. The points in K of $\mathscr{A}_n$ form the group $Pl(n, K)$ which is of dimension $n^2 - 1$. Then the structure of the projection map $p_S : V_S \to U_S$, developed in the previous theorem, easily implies the statements of this corollary.

It is clear now that to continue this investigation, at least for the free algebra S, one should study the complement of U_S in the variety of semisimple representation. This means to study the equivalence classes of reducible representations. This can be done to a certain extent, giving rise to a resonably precise description of this complement, we will not insist on this problem leaving it now to a future exposition of this subject.

REFERENCES

1. S. A. Amitsur, *Embeddings in matrix rings*, Pacific J. Math. **36** (1971), 21–29.

2. M. Artin, *On Azumaya algebras and finite dimensional representations of rings*, J. Algebra **11** (1969), 532–563.

3. E. Formanek, *Central polynomials for matrix rings*, J. Algebra **23** (1972), 129–132.

4. I. N. Herstein, *Non commutative rings*, Carus Math. Mon. 15, M.A.A., 1968.

5. C. Procesi, *Rings with polynomial identities*, M. Dekker, 1973.

6. Širšov, *On rings with identity relations*, Mat. Sb. N.S. 43 (85) (1957), 277–283 (in Russian).

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